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Certainty and Domain-Independence in the Sciences of Complexity: a Critique of James Franklin's Account of Formal Science

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James Franklin has argued that the formal, mathematical sciences of complexity — network theory, information theory, game theory, control theory, etc. — have a methodology that is different from the methodology of the natural sciences, and which can result in a knowledge of physical systems that has the epistemic character of deductive mathematical knowledge. I evaluate Franklin's arguments in light of realistic examples of mathematical modelling and conclude that, in general, the formal sciences are no more able to guarantee certainty than the natural sciences. Yet the formal sciences are characterized by a 'domain-independence' that is philosophically interesting, and I argue that it is this property that Franklin actually employs to distinguish the formal from the natural sciences. I use Einstein's 'principle'/'constructive' theory distinction to contrast the domain-independence of physical theories with the domain-independence of formal mathematical theories, and show how both kinds of domain-independence function to generate the domain-independence that is observed in the complex systems sciences. © 1999 Elsevier Science Ltd. All rights reserved.

1. Introduction

> '... Hammond's project,' Malcolm said, 'is another apparently simple system — animals within a zoo environment — that will eventually show unpredictable behavior'.
> 'You know this because of... '
> 'Theory', Malcolm said.
> 'But hadn't you better see the island, to see what he's actually done?'

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> 'No. That is quite unnecessary. The details don't matter. Theory tells me that the island will quickly proceed to behave in unpredictable fashion.'
> 'And you're confident of your theory.'
> 'Oh, yes', Malcolm said. 'Totally confident.' He sat back in the chair. 'There is a problem with that island. It is an accident waiting to happen.'

The selection is from Michael Crichton's best-selling novel *Jurassic Park* (Crichton, 1990, p. 76). Ian Malcolm is a chaos theorist, a member of a team of scientists assembled by developer John Hammond to evaluate the safety and stability of his new prehistoric theme park. Jeff Goldblum plays Ian Malcolm in the movie version. Malcolm's prediction concerning the instability of the island ecosystem is borne out, with deadly consequences for most of the secondary characters in the story.

In recent years there has been an explosion of interest in complexity and complex systems in a wide range of mathematical, natural and social sciences. Were Crichton to write Jurassic Park today he would probably have identified Malcolm as a 'complexity theorist', a specialist in a variety of mathematical disciplines employable in the service of the scientific study of complex systems, such as information theory, network theory, catastrophe theory, self-organization theory, nonlinear dynamics, etc. My interest in Crichton's novel is not with chaos or complexity theory *per se*, but with the nature of the science — 'formal science' seems an appropriate description — that is practiced by those, like Ian Malcolm, who claim to have a knowledge of the world acquired not through the conventional (fallible, inductive) methods of natural science, but rather through the formal, deductive methods of the mathematical disciplines.

To illustrate, consider the contrast between Malcolm and the other scientists in the team sent to investigate Jurassic Park. The experts on prehistoric fauna and flora, Alan Grant and Ellie Sattler, are excited by the prospect of having their theoretical speculations confirmed or disconfirmed through direct observation. Are dinosaurs warm-blooded or cold-blooded, do they run like birds or like lizards, do they hunt alone or in groups? Grant and Sattler are models of the traditional natural scientist. One can almost see the classical inductive reasoning (or Bayesian conditionalization — pick your favorite theory of scientific methodology) grinding away in their heads as they observe, for the first time and with their own eyes, the subjects of their chosen science.

Malcolm, on the other hand, is not interested in the details of dinosaur physiology or behaviour. Yet he is confident that the island ecosystem will exhibit some form of surprising, unpredictable behaviour that was not planned for, that disaster is inevitable, and all this on the basis of a formal analysis of a highly idealized (one must assume, given Malcolm's indifference to biological detail) mathematical model of the island ecosystem. As traditional science goes, a prediction that *something* unexpected is going to happen is pretty wishy-washy. But the novel grants that Malcolm is right, and that Malcolm *knew* that he was right. The island *was* somehow fated to exhibit unpredictable behaviour, and Malcolm's computer model did accurately represent (on a global scale at least) the dynamics of the island ecosystem. Malcolm's model captured certain structural features of the system that *necessitated* a certain qualitative (in this case, nonlinear or chaotic) behaviour.

Malcolm is a fictional character, but let us consider him seriously for a moment. In his own words, Malcolm is not a pure mathematician, but a 'chaotician', a *scientist* who studies complex phenomena through the lens of his chosen discipline, nonlinear dynamics. But if Malcolm is really a scientist, then what is Malcolm's science a science of? Grant and Sattler study extinct life-forms, but what does Malcolm study? Nonlinear dynamics is not a science of biological organisms, or atoms and molecules, or any restricted class of natural systems. It is, rather, a formal theory of a certain class of abstract mathematical objects or structures. The knowledge that Malcolm brings to an empirical investigation is a knowledge of these structures, and of facts relating to and deducible from these structures. Jeff Goldblum could have been parachuted into any number of different sci-fi disaster movies with different scientific settings — as the scientist studying nonlinear dynamics of brain processes, or global climate change, or patterns in signals from outerspace — with little or no change to the nature of the contribution he would make to the problem at hand. He is *ex hypothesi* an expert on complexity, wherever it may be found. But what kind of a science is this?

Crichton's fictional portrayal of the application of a formal, complex systems science to real-world phenomena is stripped of all realistic detail, but for our purposes this is a virtue, for it presents a simple conception of the epistemology and methodology of the formal sciences which can focus discussion. The features of this conception are:

(1) the independence of the content of formal science from the details of the material constitution of the systems under study,
(2) the emphasis on formal structures and relations of necessity within these structures,
(3) the claim that such relations of necessity can be true of real-world systems, and
(4) the claim that, at least in certain cases, we can know with a kind of deductive certainty that such relations do indeed hold of particular real-world systems.

In a provocative article on the nature of formal science entitled *The Formal Sciences Discover the Philosophers' Stone*, James Franklin (1994) argues that, in fact, the above four points form the methodological core of all formal science. On Franklin's view, the kind of science practised by Ian Malcolm is not only a conceptual possibility, but a model for the way all formal science is actually practised. This is a striking claim, worthy of consideration if only to figure out what would motivate anyone to believe it.

In this paper I review and evaluate Franklin's conception of formal science. I show that Franklin's radical epistemological claim — that the formal sciences allow the discernment of facts about the empirical world that have the certainty of mathematical knowledge — is supported only by the most simplistic applications of formal science, and is not applicable to real-world examples of mathematical modelling of physical systems. Though his characterization of formal science as a science of mathematical structures may be appropriate in some cases, I argue that many of the sciences that Franklin calls 'formal' make essential reference to physical principles that are contingently, not necessarily, true.

2. Science Without the Sweat?

To motivate Franklin's conception of formal science we shall borrow Ian Malcolm for a while and indulge in a little creative fiction of our own. Let us update Malcolm so that he is an expert not only in chaos theory, but in a wide range of formal sciences, from game theory to information theory to catastrophe theory.[1] And let us grant him the ability to make elaborate mathematical calculations on the spot, in his head.

Our story begins when Ian Malcolm, Super Complexity Theorist, is invited to a potluck dinner hosted by one of his university colleagues. In attendance are a number of Natural Scientists. Much wine and cheese is consumed, and the crowd breaks up into small groups, each concerned with their own particular, vexing research problems.

2.1. On the Stairs with Jill

Malcolm walks over to the stairwell and sees his host's young daughter Jill sitting at the bottom of the stairs holding a plastic toy of some kind, deep in concentration.

'What have you got there?', he asks.

'It's a puzzle that my dad gave me', Jill replies. She hands Malcolm a flat board with a series of ridges in its surface, along which a small bead can roll. The ridges connect four coloured areas. 'The narrow ones are the mainland and the two oval-shaped ones are islands in the middle of a river', Jill explains. 'You have to find

[1]Franklin presents a rather long but not exhaustive list of disciplines that he wants to include in the category of formal science. These include post-World War II systems and engineering sciences such as operations research, control theory, cybernetics, information theory and game theory; computer-related disciplines like computational complexity theory, computer simulation and theoretical computer science; complexity sciences such as the theory of cellular automata, self-organizing systems, and non-equilibrium thermodynamics; mathematical branches of so-called non-physical science, such as mathematical economics or mathematical ecology; and several branches of theoretical physics, including statistical mechanics, fluid dynamics and nonlinear physics (Franklin, 1994, pp. 515–21). We shall discuss Franklin's criteria for identifying the formal sciences below.

a way to roll the bead across all seven bridges without crossing any twice. I haven't figured it out yet.' (Fig. 1)

Always anxious to try his hand at a brainteaser, Malcolm rolls the bead around the board, looking for a path across all seven bridges. He pauses for a moment, then his eyes widen. 'Your daddy is a bit of a trickster, Jill', he says. 'You can't win this game.'

'Why not?', she asks.

'If you enter and leave a land area', Malcolm explains, 'you use up two of the bridges. That means that, except for the two chosen for the start and finish, all the land areas have to have an *even* number of bridges leaving them, or there will necessarily be bridges left over, no matter what route is chosen. But in the puzzle all four land areas have an *odd* number of bridges leaving them, so a path going across all bridges exactly once is impossible.'

Jill isn't sure she follows Malcolm's reasoning, but she grabs the puzzle and bounds up the stairs in search of her father.

2.2. In the Kitchen with Rob

Malcolm walks into the kitchen to get a bottle opener. He finds Rob, a physics student, crouched beside the sink, watching droplets of water fall from the end of the faucet. Rob says he's noticed an interesting phenomenon. He's been recording the times between water droplets and can find no discernible pattern. He suspects that the droplet times are distributed completely randomly, and is curious about the details of the physical process of drop formation that would cause such random behaviour.

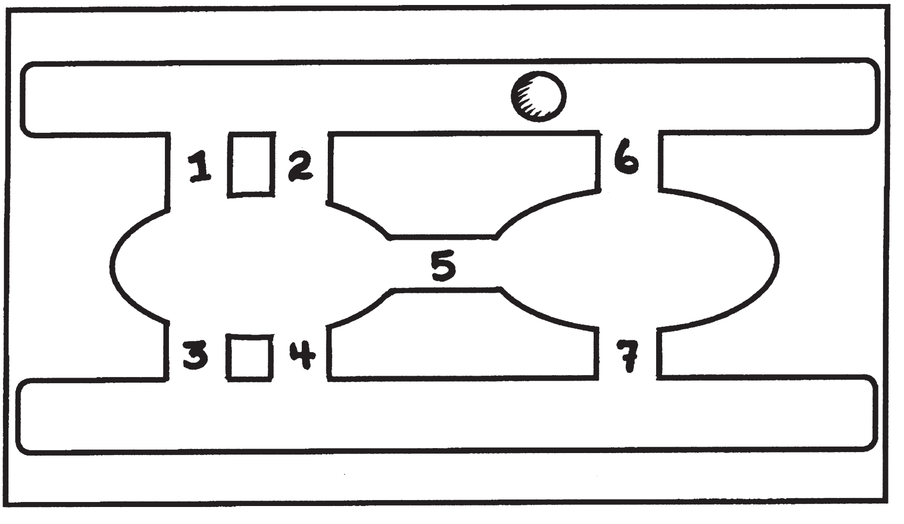

Fig. 1. Jill's bead game. The objective is to roll the bead across all seven bridges without crossing any bridge twice.

Malcolm asks to see the record of droplet times, and Rob hands him a sheet of paper with a long list of numbers. Malcolm looks at the list for a while, rubs his chin, then asks Rob whether he's noticed a period-doubling pattern of droplet times at lower flow rates. Rob admits he's never paid attention to what happens at lower flow rates, and turns the faucet knob down a notch. A pattern of times emerges that repeats every eight drops. Rob turns it down a bit more, and a four-drop pattern appears. Once again, and a two-drop pattern is heard. A final turn and the droplets assume a regular, single-period beat.

'Now watch', says Malcolm, and he turns up the flow rate past the point where the random drop sequence was observed. 'I'll bet you get a three-drop pattern up here', he says. A three-drop pattern is heard, and Rob is shocked.

'How did you know those droplet patterns would be there?', he asks Malcolm. 'And what kind of physical process would produce such complex behaviour? It must be frightfully complicated.'

'Oh no', replies Malcolm, 'I'm sure it's quite simple.' He explains that the droplet times for the 'random' sequence weren't really random at all, but only 'chaotic'. 'There are correlations between successive drop times, but you won't notice them unless you plot the points as a two-dimensional scatter plot, with time t_n plotted on the *x*-axis and time t_{n+1} plotted on the *y*-axis. You get a kind of parabolic ribbon structure when you plot the times this way, which indicates a quadratic relationship between successive times. Chaotic systems of this type have a characteristic period-doubling route to chaos, and intermittent windows between chaotic regions where the periods are odd-numbered.'

The complex dynamics of the system emerges from a simple, nonlinear, deterministic relationship between a small number of variables, explains Malcolm. 'I suspect you could model a system like this with a simple mass-on-a-spring arrangement, letting the mass be a function of time. When a droplet fills up with water it will stretch the column of water that secures it to the faucet. When it breaks off, the column will recoil, and the time for the next droplet to form and break off will depend on the flow rate and whether the column is on the up-swing or the down-swing of the recoil when the droplet gets heavy again. That's probably where your nonlinearity enters.'

Rob is thankful for Malcolm's help, and grateful that he doesn't have to bother with the detailed physics of surface tension and fluid flow to explain this curious phenomenon.

2.3. On the Patio with Linda, Harry and John

Malcolm is invited to sit down for a drink with Linda, Harry and John, who are ecologists working on forestry management problems. They tell Malcolm about their current project, which is to develop a mathematical model of spruce budworm infestations in the spruce and fir forests of eastern Canada and the northeastern United States. These forests have periodically been subject to ravages by the spruce budworm caterpillar. For a number of years, a given patch of forest is seen to grow with hardly any budworm in evidence. When the trees have reached a certain level of maturity there is an explosive increase in the number of budworms and they begin to defoliate the trees. When a stand of mature trees have been sufficiently denuded over several consecutive years, they wither and die. The budworm population within the patch can no longer be sustained since its food supply becomes scarce. Their numbers decrease and then quite suddenly collapse to a low subsistence level. But the forest canopy has been opened up, which allows new seedlings to grow. The forest renews itself and a new cycle begins, which eventually leads to another outbreak of insects in about thirty to seventy years.

The ecologists explain to Malcolm that they've just finished work on a mathematical model of the spruce budworm cycle that relates budworm density (*B*) to tree branch surface area (*S*) and the percentage of foliage on the trees (*E*). Linda hands Malcolm a sheet of paper with the following equations written on it.

$$\frac{dB}{dt} = \alpha_1 B\left[1 - \frac{(\alpha_3 + E^2)}{\alpha_2 S E^2}\right] - \frac{\alpha_4 B^2}{(\alpha_5 S^2 + B^2)}$$

$$\frac{dS}{dt} = \alpha_6 S\left[1 - \frac{\alpha_7 S}{\alpha_8 E}\right]$$

$$\frac{dE}{dt} = \alpha_9 E\left[1 - \frac{E}{\alpha_7}\right] - \frac{\alpha_{10} B E^2}{S(\alpha_3 + E^2)}$$

'All those undefined parameters, the αs, represent various intrinsic growth rates and predation rates', says John. 'The model captures all the basic qualitative features of the outbreak pattern, even the sudden jumps in budworm population.'

'What we want to do', says Harry, 'is find a way of stabilizing *B* at a low level. We figure there has to be some combination of these parameters that will do the trick, but there are so many variables that we've just about given up hope of finding one.'

'Hmm... ' mutters Malcolm, pulling a pen out of his shirt pocket. 'You want to set the right hand side to zero, right? That'll give you a big long equation in *E* and *S*, but you can eliminate *S*, and that'll give you this as the equilibrium condition, right?' he says, writing down the equation.

$$\bar{B} = \frac{-\alpha_8\alpha_9}{\alpha_7^2\alpha_{10}(\bar{E}^3 - \alpha_7\bar{E}^2 + \alpha_3\bar{E} + \alpha_3\alpha_7)}$$

'That's right!' says Linda. 'But we don't know how to choose the parameters that will ensure that the equilibrium is stable.'

Malcolm sighs. 'You can't do it', he says. 'Your stable equilibria lie on the upper and lower folds of a dual cusp catastrophe surface, and the unstable equilibria lie within the cusp region. You need to choose your αs so that the system stays out of that cusp region, but there aren't any physically realizable values for the αs that will do the trick. You can't control this system.'

'Hunh?', says Linda. 'Can you run that by us again?'

Malcolm explains that you can write the equation for $\bar{B}$, the steady-state condition for budworm density, as a monic with no quadratic term by introducing a new variable:

$$y = \bar{B} - \frac{\alpha_2\alpha_8\bar{E}^3}{3\alpha_7(\alpha_3 + \bar{E}^2)}$$

After a bit of manipulation, you can show that y satisfies the cubic equation

$$-(y^3 + t_1 y + t_2) = 0$$

which a catastrophe theorist will recognize as the equilibrium equation for the standard form of the *cusp catastrophe*. The parameters t_1 and t_2 are given in terms of the original system parameters as

$$t_1 = \frac{-\alpha_8\bar{E}^2}{\alpha_7^2}\left[\frac{\alpha_2^2\alpha_8\bar{E}^4}{3(\alpha_3 + \bar{E}^2)^2} - \frac{\alpha_2\alpha_4\alpha_7\bar{E}}{\alpha_1(\alpha_3 + \bar{E}^2)} - \alpha_5\alpha_8\right]$$

$$t_2 = \frac{-\alpha_2\alpha_8^2\bar{E}^5}{9\alpha_7^3(\alpha_3 + \bar{E}^2)}\left[\frac{2\alpha_2^2\alpha_8\bar{E}^4}{3(\alpha_3 + \bar{E}^2)^2} - \frac{3\alpha_2\alpha_4\alpha_7\bar{E}}{\alpha_1(\alpha_3 + \bar{E}^2)} + 6\alpha_5\alpha_8\right]$$

Geometrically, the system can be represented as a three-dimensional system, where the behavioural variable, y, is a function of two control variables, t_1 and t_2. The cusp geometry gives you generic stability conditions for systems with two inputs and one output. Malcolm sketches a diagram showing the cusp catastrophe surface (Fig. 2).

'What you want to do is manipulate the αs to stabilize the budworm density on the lower sheet of the manifold', Malcolm explains, 'but you have to stay out of the shaded cusp region, because it's unstable. The equation for this region is simple:

$$4t_1^3 + 27t_2^2 \geq 0$$

This is the necessary condition in order to be able to stabilize the budworm densities at a low level. But if you look carefully at the physically realizable values of the α parameters, you'll see that there is no combination that will satisfy this condition.'

The ecologists are stunned. 'What does this mean?' asks Harry. 'Is there no way to avoid these outbreaks?'

'All it means it that no amount of "knob-twisting" with the α parameters will suffice to control the system', replies Malcolm. 'That doesn't mean the system can't be controlled, just that any effective scheme will have to be based on more sophisticated methods of dynamic control.'

Malcolm excuses himself from the table, wishes everyone a good evening and drives home. Along the way he notices that the timing between red, green and yellow lights at a number of traffic intersections is not quite optimal, given the joint goal of maximizing traffic throughflow and minimizing energy wasted through

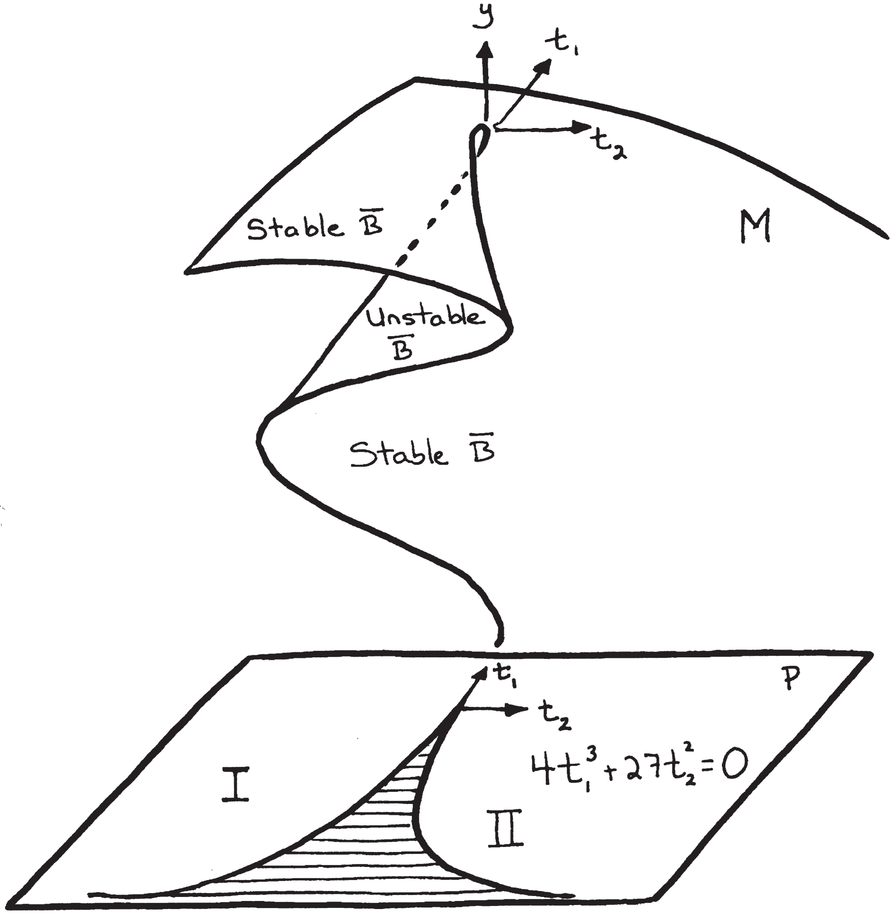

Fig. 2. Malcolm's drawing of the catastrophe cusp manifold for spruce budworm outbreaks.

starting and stopping. He makes a mental note to call the city transportation authorities in the morning.

3. Franklin's Account of Formal Science

Readers may recognize one or two of the applications of formal science described above. The first is widely known as the 'Königsberg Bridges Problem'. The citizens of Königsberg noticed that it seemed impossible to walk across all seven bridges over the river Pregel without walking across at least one of them twice. Leonhard Euler proved their conjecture correct, using the simple reasoning described. Euler's proof is now regarded as the first study in the topology of networks. James Franklin uses this example specifically to illustrate the general features of his account of formal science.

The second example is derived from Robert Shaw's classic treatment of chaotic dynamics in a dripping faucet.[2] The catastrophe-theoretic analysis of the spruce budworm outbreak is familiar to theoretical ecologists,[3] though the proof that the system cannot be stabilized by parameter 'knob turning' is certainly less familiar.[4] I introduce these examples as an aid to explicating Franklin's account of formal science and to focus later discussion.

Franklin wants us to consider the nature of the contribution that a person trained in network theory, or nonlinear dynamics, or catastrophe theory, can make to our understanding of physical phenomena. In Franklin (1994), his primary concern is with the epistemic character of the knowledge of physical phenomena acquired through formal means, and the method by which this knowledge is obtained. In this section we will consider two elements of the epistemic character of formal knowledge that Franklin identifies: (i) domain-independence, and (ii) mathematical certainty.

The reasoning that Malcolm applies in each of the above cases is, in a strong sense, domain-independent. In each case the system under investigation is recognized to have a formal structure that can be captured in mathematical form. Malcolm then brings his mathematical knowledge to bear on the system and deduces certain mathematical facts that are physically interpretable, and relevant to the scientific problem at hand. But in each case the mathematical reasoning involved is quite general, in that it is not tied to the particular material or ontological constitution of the system in question. The impossibility of crossing all seven bridges without crossing any twice is a restriction on *any* conceivable system with the appropriate network topology. Similarly, the period-doubling route chaos is a characteristic of any mapping with quadratic maxima, and the cusp catastrophe is a generic stability feature of any two-input single-output system governed by a point attractor. One can imagine the same analyses being applied to systems of radically different ontological makeup.

Second, the insights into the physical phenomena studied in the above examples appear to have the character of mathematical or deductive certainty. Once Malcolm realizes that Jill's game has a certain network structure, he is able to say, with certainty, that there is no solution path. On the basis of the correlations observed in Rob's water droplet data, Malcolm knows with the utmost confidence that the pattern is not random, and that it is caused by a characteristic period-doubling sequence of bifurcations. Given the equations that describe the spruce budworm outbreak, Malcolm is able to say without hesitation that no amount of parameter-twiddling will stabilize the system. Franklin believes that the knowledge of physical systems contributed by the formal sciences can have the character and certainty of

[2]Shaw's experiment is described in Gleick (1987). For details of the analysis see Martien *et al.* (1985); Yepez (1989).

[3]See Ludwig *et al.* (1978).

[4]See Casti (1982) for the original paper. This example is also discussed in Casti (1992).

mathematical knowledge. Consequently, this knowledge will never be rendered obsolete by new scientific discoveries. The formal sciences have, in a real sense, discovered the 'philosophers' stone':

> ... knowledge in the formal sciences, with its proofs of network flows... and the like, gives every appearance of having achieved the philosophers' stone; a method of transmuting opinion about the base and contingent beings of this world into the necessary knowledge of pure reason. (Franklin, 1994, p. 513)

The formal sciences may appeal, Franklin continues, to

> the many who feel that philosophers of science have chatted on to one another sufficiently about theory change, realism, induction, sociology, and so on, while real science has been producing a huge and diverse body of knowledge to which all that is totally irrelevant. (p. 513)

Precisely how are we to understand the claim that formal knowledge has the character of mathematical certainty? Granting that mathematical reasoning about mathematical objects has a deductive character, in order for this reasoning to carry over directly to a physical system, must we not *already* be certain that a given physical system actually instantiates the appropriate formal structure?

Franklin agrees that establishing the formal structure of a physical system is necessary for our knowledge of the physical system to take on the character of mathematical knowledge. However, he argues, in many cases this is achievable. In uncomplicated cases like the Königsberg bridge problem, the formal structure is readily apparent to our perceptual faculties; we simply *look and see* how many land masses there are and how many bridges there are, and how they are connected.

How do we know that we aren't mistaken in our perceptions? Never, says Franklin, if knowledge requires 'absolute' certainty — there is always the chance that one is hallucinating, or that one of the bridges is a hologram projected by an alien space-ship, or an evil demon is messing with one's head. But this kind of uncertainty attends *all* perceptual knowledge. Rather, our knowledge of the network structure of the bridges has 'practical' certainty, the certainty we have with respect to ordinary perceptual judgments made under ordinary viewing conditions, such as the judgment that my coffee cup is empty, or that my computer is sitting on top of my desk rather than beneath it. The assumption of 'practical certainty' is required even for traditionally acquired mathematical knowledge, since the certainty obtained by following a proof of a theorem presupposes that one hasn't misread a step or been deceived at some stage in the proof.

Franklin makes much of the role of the computer in the methodology of the formal sciences. It is also possible, Franklin reminds us, to solve the Königsberg bridges problem without any mathematical ingenuity at all, by simply checking by computer whether all the possible paths which do not go over any bridge twice (there are less than a thousand of them) go over all bridges once. The result is exactly the same, and demonstrates the same impossibility with the same necessity as the earlier reasoning. Notice also that though we may not be able to 'survey', through direct observation, the network structure of more complicated cases, we *can* survey the simple cases, and we can survey the correctness of the steps in the computer algorithm which performs the calculation for the complex cases. The computer is able to extend the practical certainty acquired through direct perception of simple cases to more complex cases because the computer program is itself a formal system that transforms inputs into outputs through a chain of necessary entailments.

At this point it becomes clear why Franklin chooses to call nonlinear dynamics or network theory a science, rather than a branch of applied mathematics. Franklin believes that physical systems can instantiate mathematical structures of various kinds, and that mathematical structures are proper objects of sensory experience. In this he sides with philosophers of mathematics of the structuralist school (e.g. Resnick, 1981), who regard mathematics as a science of 'structures' or 'patterns', and who

> agree that the objects of mathematics should not be interpreted in a Platonist sense, but should be reinterpreted as things available through ordinary sense perception. (Franklin, 1994, p. 523)

Formal science is science because it makes possible a kind of knowledge of physical systems that, like the knowledge acquired in natural science, is grounded in perception.

On the other hand, the epistemic character of formal science is different from that of natural science because the exclusive use of mathematical reasoning 'removes, through proof, the further source of uncertainty found in the physical and social sciences, arising from the uncertainty of inductive reasoning and of theorizing' (p. 528).

4. Reality Check

The methodology of the formal sciences is summarized by Franklin as follows:

(1) There are connections between the parts of the system being studied, which can be reasoned about in purely logical [or mathematical] terms.
(2) The complexity is, in small cases, surveyable. That is, one can have practical certainty by direct observation of the local structure. Any uncertainty is limited to the mere theoretical uncertainty one has about even the best sense knowledge.
(3) Hence the necessity [of the reasoning among the connections] translates into practical certainty.
(4) Computer checking can extend the practical certainty to much larger cases. (Points 1–4 from Franklin, 1994, p. 529)

It is unfortunate that Franklin gives no examples of applications of formal science other than the Königsberg bridge example.[5] A proposal that purports to draw a principled distinction between the category of 'natural' science and that of 'formal' science, and that claims to give a characterization of the methodology of *all* formal sciences, requires at least some demonstration that it applies to more than the single, simple case chosen to illustrate it.

In the absence of examples provided by Franklin, let us consider the two additional examples introduced above, and see whether they fit Franklin's model. In the first example Malcolm uses chaos theory to discern a number of interesting features of the dynamics of a dripping faucet. The drip times are analyzed and correlations are observed which, when plotted in the appropriate phase space, reveal an inverted parabolic structure (which would, upon closer analysis, reveal a fractal geometry). From this structure Malcolm is able to infer that the dynamics of the system is describable by the period-doubling route to chaos. He then makes a couple of predictions concerning the drip patterns that will be heard at flow rates above and below the chaotic region, which are confirmed. He later offers a hypothesis concerning the mechanism which might give rise to the observed dynamics.

Now, are there 'connections between the parts of the system being studied which can be reasoned about in purely logical terms'? Yes, if we start the process of inference from the observed data and follow the steps leading to the period-doubling pattern. But in reality one needs fairly precise time measurements in order to discern the correlation structure that actually governs the system dynamics. In our fictitious example we imagine Rob with a stop-watch making measurements, but one needs a laboratory setup with accurate measuring instruments to record data which actually reveal the underlying attractor structure.[6] But this point does not significantly conflict with Franklin's account if one grants that there is *some* way of acquiring data that will resolve the attractor structure. If the attractor has the characteristic inverted hump structure, then the inference to a period-doubling route to chaos is automatic.

Once Malcolm is secure in his knowledge that there is an underlying period-doubling dynamics present in the dripping faucet system, can he be as secure in his prediction that one will actually hear a periodic pattern of droplets at lower flow

[5]He does discuss one other example from computer science, concerning attempts to write proofs that a program is error-free ('program verification'), but his discussion of this example focuses on the question of whether mathematical properties are genuinely predictable of physical systems at all. Franklin appeals to the structuralist tradition in the philosophy of mathematics in support of this view, but does not offer any independent defense of structuralism in Franklin (1994). For an earlier discussion where Franklin tries to make the case that mathematical properties are genuinely predicable of physical systems, see Franklin (1989). For the sake of argument I have granted the premise that some form of structuralism is true, and hence that it makes sense to talk about a science that studies the mathematical properties of real physical systems. My criticisms of Franklin's account of formal science, however, do not turn on this premise.

[6]I performed this experiment in an undergraduate physics lab, and for my particular setup, period-doubling was observed at about 9 drops/sec and chaos set in around 13 drops/sec. It took a laser and a microcomputer to record the time intervals with sufficient accuracy to observe the ribbon-like structure of the underlying chaotic attractor.

rates? In a realistic experiment laboratory equipment may be required to isolate the system from external influences and regulate the flow rate with sufficient precision in order to observe predicted patterns of behaviour. Thus, Malcolm could not be secure in his prediction regarding the actual behaviour of the dripping faucet system. In our hypothetical example he just got lucky.

Nor can Malcolm be certain about his proposed mechanism for generating the nonlinearities in the system. The interaction between the spring-like dynamics of the water column and the increasing mass of the droplet is one plausible mechanism (it has the right 'stretch and fold' character of all chaotic systems[7]), but it is not the only conceivable one. At best, Malcolm could be certain that *some* kind of stretch-and-fold dynamics is operating somewhere in the system. Such knowledge can be an enormous aid in mathematical modelling, and a simple mass-on-a-spring model may capture the dynamics quite well. But it in no way guarantees that one has isolated *the* causal mechanism that is responsible for the dynamics in this particular case.

Let us now consider the spruce budworm example. Linda, Harry and John had already developed a mathematical model for a forest patch. Malcolm was able to perform a number of formal operations on this model, reducing it to a form which allowed it to be analyzed in terms of catastrophe theory. Once the abstract form of the model was given, the impossibility of keeping the budworm density on the lower sheet of the cusp and out of the unstable cusp region followed deductively. This is clearly important information for anyone committed to the adequacy and completeness of the initial model, but it should be obvious that the construction of such models in ecology, economics, or any other area where fundamental laws are rare or non-existent (and even reliable empirical generalizations are hard to come by), is as much an art as it is a science. Simplifying assumptions and idealizations are essential to the construction of such models, and even when a good balance is achieved between empirical adequacy and analytic or computational tractability, most modellers are aware that they are dealing with mathematical cartoons of real-world phenomena, not the phenomena themselves. Malcolm's claim that the budworm outbreaks can't be controlled is entirely contingent on the acceptance of a highly idealized model of the phenomena.

As Casti (1982) states, what is really interesting about the catastrophe analysis of the model is that it shows that the number of physically meaningful parameters in a problem may be very different from the number of *mathematical* parameters needed to address the question of interest. In our example we had ten physically important parameters (the αs) given as part of the original problem statement; however, upon carrying out the elementary analysis of the equilibrium equation for B, it turned out that the real question of interest regarding the possibility of

[7]That is, there is a mechanism that tries to increase the value of a variable without bounds, ensuring that neighbouring points in state space diverge exponentially, and another that maps the variable back onto a fixed interval in its state space, resulting in chaotic motion within the interval.

regulating the budworm density by parametric variation came down to the interrelationship between the two mathematical parameters t_1 and t_2. Each of these parameters is a complicated algebraic combination of all ten of the physical parameters. It is very unlikely that any amount of guesswork would find that this combination of the α parameters — and no other — is the relevant combination for addressing the question of budworm outbreaks. The empirical significance of the catastrophe analysis is not that it rules out the possibility of managing budworm outbreaks, but that it gives us insight into what does and doesn't count in the analysis of the system in question.

This example illustrates a general problem with Franklin's account of methodology in the formal sciences. On Franklin's account, for knowledge of a formal structure to count as knowledge of a physical system, one must establish that the physical system instantiates the formal structure. But in the majority of realistic modelling situations, the models involved are simplified abstractions of the real system, and strict isomorphism between the model and the physical system is impossible to establish. Insofar as Franklin's account *requires* that such an isomorphism obtain, it rules out of consideration all but the most simple and contrived models, such as the network model for the Königsberg bridges problem.

But as a consequence of this strict requirement of isomorphism, Franklin's account makes it difficult to appreciate the diverse ways that real applications of formal science *can* contribute to our understanding of a physical problem. In both the chaos theory and catastrophe theory examples, the complex dynamics of a dripping faucet and a forest patch were found to depend on only a few parameters, effectively reducing a complex multi-dimensional system to a simple, low-dimensional system with the same qualitative dynamics as the original. Such analyses can yield significant insight into the behaviour of the original system, but they do not depend on the establishment of the structural identity of a real system and a formal system.

This is not to say that reducing the dimensionality of a problem, or constructing formal analogies that mimic the dynamics of a natural system, is the only way that formal science can contribute to our understanding of a physical system. It is to say, rather, that there are *many* ways that formal methods and formal models are used in science, and many (if not most) of these do not require that the formal model be structurally identical to a natural system.

5. A More Charitable Interpretation

At this point we should pause and consider whether we have interpreted Franklin correctly, for it seems too obvious a fact that the formal sciences do not always operate with physical systems that are known to instantiate a formal structure. Does his account of formal knowledge really require such a close relationship between models and the world? The emphasis that he places on 'practical certainty' would seem to indicate that he does require it, but there is evidence in his article which supports a more charitable and plausible interpretation.

Franklin addresses the model–reality gap problem in the last section of his paper, where he considers the role of experimentation in the formal sciences:

> Real certainty for armchair work — surely this is too rosy a picture of the formal sciences? If it were right, it ought to be possible to issue real-world predictions by computer, without needing to do any experiments. Anyone who has worked in applied mathematics knows it is rarely like this. It is well known that fitting a realistic mathematical model to actual data is in general difficult. Sometimes, as in meteorology and macroeconomics, it is virtually impossible... Everyone agrees that formal work can proceed with the usual necessity of mathematics, provided one keeps within the model. The important point is that there is wide variability in the certainty in deciding whether the real world has the structure described by the model. The model–reality gap may be wide or narrow. (Franklin, 1994, p. 532)

Franklin even admits that his examples are tailored to fit his methodological model:

> The examples above were chosen near the opposite extreme, even, so it was argued, to the extent that there was no gap [between model and reality] at all. What structure a system of bridges or a computer program has is open to perceptual inspection, with the practical certainty that attends unimpeded sense perception. So all the hard work is in the mathematics, and the results are directly applicable, again with practical certainty. (p. 533)

But if the 'real certainty' characteristic of formal knowledge is applicable only to a very small class of systems, then why advertise it as a general feature of all formal science? Some insight into this question may be gained by considering several comments that Franklin makes regarding the formal status of various branches of theoretical physics. These comments suggest a different interpretation of the essential character of formal science.

> In retrospect, certain aspects of theoretical physics have a character recognizably like the formal sciences. Statistical mechanics, going back to Maxwell and Boltzmann, looks at how macroscopic properties of gases, like pressure and temperature, arise as global averages of the movements of the individual particles. The emphasis is not on details about the properties of the particles themselves, but on the transition from local to global properties. The same is true of fluid dynamics, especially in the very difficult study of turbulent fluids. The organization of the fluid flow into eddies and smoke rings is plainly not to be explained by examining the individual atoms more closely. Non-linear physics treats more generally the ways in which complicated global structures can arise from simple local interactions. (p. 521)

Franklin is contrasting theoretical speculation concerning the natures of the component parts or hypothetical constituents of a system, with the explanation of system properties and behaviours that arise as collective phenomena or as mathematical consequences of underlying dynamics. The move from microscopic to macroscopic properties in statistical mechanics proceeds in a purely formal way, and can be applied to a diverse range of systems as long as properties of systems at the microlevel relate to properties at the macrolevel in the appropriate way. Similarly, certain phenomena, such as the transition from laminar to turbulent flow in fluid dynamics, are generic properties of a certain class of nonlinear dynamical systems, and do not depend on the detailed structure of the microconstituents.

While the *existence* of these formal properties is contingent on the existence of system components of a certain kind, the relationships between formal properties remain a matter of necessity:

> Whether the kinetic theory of gases is true is a contingent fact, not easily established. But it is in fact true, and the way temperature arises from the random motion of gas particles is a matter of necessity. Though it is harder than in the case of the bridges to determine if things have the properties, there is real necessity in the *connections* of the properties. Being provable, it is a stronger necessity than nomic or Kripkean necessities. (p. 533)

In light of these comments, I offer the following reconstruction of Franklin's account of formal science:

(1) Natural systems possess formal, mathematical properties, which are deductive consequences of the natures and arrangements of the hypothetical constituents of the system.

(2) Because these formal, mathematical properties are provable, they can be known with deductive certainty *on the assumption that the hypothetical constituents of the system exist and have the natures presumed in* (1).

(3) For certain systems we *can* have practical certainty that the relevant constituents exist and possess the properties as given in (1). This practical certainty is grounded in the fact that when structural relationships *are* instantiated in physical systems, they *may* be directly accessible to perception.

(4) For many systems we cannot be certain that the assumptions necessary for the deduction of formal properties obtain, either because the system is too complex or because the assumptions are of a theoretical nature, inaccessible to the senses via direct observation. In such cases one does *not* have practical certainty about the formal properties of the system.

(5) The distinctive nature of the formal sciences is this: *they tell us what the formal, domain-independent properties of a system are or would be, given certain assumptions about the natures and arrangements of the hypothetical constituents of the system.*

(1) makes an ontological claim about the reality of mathematical properties, which Franklin defends on pp. 523–26. (2) and (3) together assert an epistemological claim about the kind of knowledge that these properties make possible; this is the main focus of Franklin's paper. (4) simply admits what we all know to be the case, and Franklin acknowledges to be so on p. 533. The only claim that is applicable to the formal sciences *as a whole* is (5), and this, I contend, is what Franklin intends as the essential feature of formal science that distinguishes it from natural science; it is what is meant by saying that formal science, like any branch of mathematics, is a science of 'relations', 'pattern' or 'structure' (Franklin, 1994, p. 525).

Franklin's article gives the impression that he regards the epistemological claim — that the formal sciences offer 'practical certainty' about real-world systems — as the central feature that distinguishes formal from natural science, but one must conclude that he simply misrepresents his position, or is not clear on his position himself. Regardless, the summary given above is the most charitable and, I believe, the most defensible formulation of Franklin's views.

6. A World Full of Structures

Franklin's account of formal science raises some interesting questions concerning the nature of formal constraints and their operation in the world. Consider once again the Königsberg bridges problem. The citizens of Königsberg could not find a path across all the bridges that did not cross one bridge twice. Why not? What prevented them from finding such a path? The natural answer is that the network structure of the bridges imposed a *formal constraint* that all paths through the network were required to satisfy. And this same network structure was responsible for Jill's frustration with the game that her father had given her. This kind of structural constraint is not universal in scope, for it applies only to systems with a given network topology, but it *is* strictly domain-independent, applicable to any conceivable type (physical, biological, artificial, social) of system.

Franklin adopts a structuralist philosophy of mathematics, a view that regards mathematical structures as real, genuine properties of physical systems. On a structuralist account, the network topology of the Königsberg bridges is a real property of *that* physical system. As one contemplates the many different kinds of formal structure that are conceivably instantiated in the world, this view naturally leads to an expansion and diversification of the formal 'ontology' of the world. The world appears densely populated with formal structures that constrain phenomena in a myriad of different ways. Beads are constrained to follow certain paths and not others in children's games. Dripping water is constrained to burst into chaotic rhythms at the turn of a faucet knob. Spruce budworm populations are constrained to explode and shrink in rapid, discontinuous jumps.

When presented in this light, a *science* of formal constraints doesn't seem so odd. Processes and events in the world are governed by physical laws of various kinds, but they are also governed by purely structural, formal constraints which operate at all spatial and temporal scales. Understanding how these formal constraints operate in the world is a legitimate scientific pursuit, and it may well have a distinctive character from that of the traditional natural and social sciences. Ian Malcolm may be a fictional character, but the traits that mark and distinguish him from his fellow natural scientists — a focus on mathematical theories and computer models; relative indifference to the details of the material constitution and causal mechanisms at work in specific natural systems; a degree of certainty about the possibility or impossibility of the occurrence of certain phenomena that is rarely observed in traditional, empirically-oriented natural science — are not fictions, but inherent characteristics of a science which specializes in formal structure.

7. Principle Theories and Formal Constraint

All this talk of structural constraints on events or processes may bring to mind the distinction introduced by Einstein between 'principle' theories and 'constructive' theories.[8] Constructive theories postulate 'hypothetical constituents' that are used to 'build up a picture of more complex phenomena out of the materials of a relatively simple formal scheme' (Einstein, 1919, p. 228). The Kinetic Theory of Gases, for instance, conceived a gas as composed of hypothetical constituents called 'atoms' or 'molecules', which were modelled as elastic spheres or point centers of force, colliding with one another and with the sides of the container which contained the gas. The aim of a constructive theory is to reduce a wide class of diverse systems to component systems of a particular kind.

'Principle' theories, on the other hand, have potentially universal application. Principle theories specify principles or laws that impose structural constraints on the interactions or processes described by lower-level constructive theories. Einstein's favorite example of a principle theory is Classical Thermodynamics, where all physical processes are stipulated to satisfy conservative (First Law) and dissipative (Second Law) constraints. Einstein regarded Newtonian Mechanics and the Special and General theories of Relativity as principle theories as well.

The constraints that principle theories impose are often described as formal or mathematical constraints on the structure of spatial and temporal events (e.g. Bub, 1973, p. 142). Thus, Newtonian mechanics imposes the inhomogeneous Galilean group as the symmetry group of free motions; Einstein's principle of relativity asserts that the symmetry group of free motion is the Poincaré group (with an associated modification in the space–time structure); and so forth.

Given the previous discussion of formal science as a science of mathematical structure, it is tempting to say that formal sciences do on the small scale what principle theories do on the large scale; that is, specify formal structures that processes and events in the world must satisfy. The traditional principle theories, one might suggest, are distinguished simply by their near-universal scope and the fundamental character of their domains.

There is a certain appeal in this view, but one must avoid conflating constraints imposed by physical principles with constraints imposed by purely mathematical or logical principles. Physical principles are contingently true, and contraints imposed by these principles have the status of contingent truths, not necessary truths. Consider the derivation of the Ideal Gas Law, $PV = nRT$, within the Kinetic Theory of Gases. The Kinetic Theory asserts that a gas is really composed of tiny

[8]For a recent and insightful discussion of the 'principle'/'constructive' theory distinction in Einstein's work, see Flores (1999).

molecules that move rapidly about, bouncing off each other and the walls of their container. By itself the molecular hypothesis is insufficient to derive any phenomenological macroscopic laws. Only after the motions of the molecules are constrained by the contingently true laws of Newtonian Mechanics (a principle theory) is it possible to derive the Ideal Gas Law. So constrained, the relationship between microstates and macrostates of a gas emerges as a purely formal relationship, with macrostates appearing as time averages of microstates. Furthermore, constraints imposed by principle theories manifest themselves in the interaction laws of constructive theories, which in turn specify the kinds of forceful interactions (mechanical, gravitational, electromagnetic, etc.) that are observed in the world. The Law of Action–Reaction, for example, is a constraint on forceful interactions (or perhaps, a constraint on what sorts of interactions are to count as true forces). Mathematical constraints typically do not manifest themselves as forceful interactions or as constraints on forceful interactions. The little bead in Jill's Königsberg bridges game was not *forcefully* prevented from following a path which crossed all the bridges without crossing any one twice.

This distinction between formal and physical constraints is important, for it requires us to distinguish two different kinds of domain-independence. A formal theory in the strict mathematical sense will be domain-independent because the theory only makes claims about the formal properties of a mathematical or logical structure. The theorems of such theories, such as network topology or graph theory, are literally not *about* physical systems at all. A physical theory may be domain-independent in a different sense. Principle theories, for example, state physical principles and general laws that are postulated to apply to all physical processes, interactions or systems, without reference to specific causal mechanisms at the 'ground' level. Domain-independence results from the fact that a large, potentially universal class of phenomena are constrained by the principles of the theory. In this case the theory has a physical domain, but the domain is so large that it cuts across conventionally defined scientific domains.

Franklin doesn't acknowledge these two different kinds of domain-independence in his account of formal science, but he should, because some of the sciences that he wishes to call 'formal' are really physical theories whose domain-independence is of the latter variety. Consider the following two 'domain-independent' claims:

> (A) There is no path through a graph with an odd number of nodes which does not cross at least one node twice.
> (B) The ratio of the magnitude of indirect to direct flows in a network increases with increasing (a) system size (number of components), (b) system connectivity (density of interactions), (c) compartment storage (flow impedance), (d) feedback and nonfeedback cycling, and (e) strength of direct flows. In fact, as a network becomes larger and more complex, the contribution of the indirect flows tends to exceed the contribution of the direct flows.

(A) is a theorem of graph theory, or 'network topology'. It is a purely mathematical result. (B) is a theorem of *network ecology*, a subdiscipline within theoretical ecology that studies the network structure of complex ecological systems. The result given in (B) is known as the Dominance of Indirect Effects (Higashi and Patten, 1989). It asserts that as a network grows in complexity, indirect feedback effects will come to dominate the activity of any given node in the network. But (B) is not a purely mathematical result. The statement of the result makes essential reference to 'flows', 'cycling', and 'interactions'. The network that is being described in (2) is a *physical* network of flows of material or energetic substance. In order to derive (2) one needs to assume that every transfer is subject to mass-balance, energy conservation and energy dissipation constraints, which are contingent physical constraints ('principle theory' constraints, the theory in this case being Thermodynamics). The Dominance of Indirect Effects is a physical hypothesis which, if true, is applicable to systems as diverse as computer networks, neural networks, cellular metabolism, economic systems and ecological systems. (B) is domain-independent in the physical sense described above, not in the purely formal, mathematical sense; it has a physical domain, but the domain is so broad that it cuts across traditional scientific boundaries.

Franklin's long list of 'formal sciences' is a heterogeneous mixture of mathematical and physical theories that exhibit different kinds of domain-independence. The field of cellular automata may be a formal science in the strict mathematical sense, but theories of self-organization and nonequilibrium thermodynamics, such as Prigogine's theory of dissipative structures and 'order through fluctuations' (Prigogine, 1980), certainly are not. Even within a field one can distinguish the different kinds of domain-independence. The theory of dynamical systems originated in classical physics, and most of the classical theorems of dynamical systems theory apply to Hamiltonian systems with potentials whose derivatives can be interpreted as real physical forces. But more general and abstract dynamical systems can also be studied (cellular automata, for example), and the theorems of this field are best seen as pieces of pure and applied mathematics.[9] A proper understanding of the complex systems sciences will require a more careful analysis of how formal and physical contraints combine to produce the complex phenomena that we observe.

8. Conclusion

In this paper I reviewed James Franklin's approach to 'formal science' as presented in his 'The Formal Sciences Discover the Philosophers' Stone' (Franklin, 1994). Despite appearances to the contrary, Franklin's emphasis on the 'practical certainty' made possible by formal science is not the feature that he is using to distinguish formal science from natural science. Rather, Franklin is using the cri-

[9]See, for example, Hirsch and Smale (1974).

terion of ‘domain-independence’ to distinguish the formal from the natural sciences. I gave a more charitable reconstruction of Franklin’s conception of formal science as a science of mathematical structure, but showed that not all of the complex systems sciences are ‘formal’ in the strict mathematical sense. Many complex systems sciences are a hybrid mix of formal and physical principles, and their domain-independence is of a different kind than is found in purely mathematical theories. More work needs to be done before we have a clear understanding of how these mathematical and physical principles interact to generate explanations of physical phenomena.

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